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The $\overline{\partial}$ operator along the leaves and Guichard's theorem for a complex simple foliation

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Abstract In (Ann Sc ENS Sér 3 4:361–380, 1887) Guichard proved that, for any holomorphic function g on $\mathbb{C}$, there exists a holomorphic function h (on $\mathbb{C}$) such that $h - h \circ \tau = g$ where τ is the translation by 1 on $\mathbb{C}$. In this note we prove an analogous of this theorem in a more general situation. Precisely, let $(M, \mathcal{F})$ be a complex simple foliation whose leaves are simply connected non compact Riemann surfaces and γ an automorphism of $\mathcal{F}$ which fixes each leaf and acts on it freely and properly. Then, the vector space $\mathcal{H}_{\mathcal{F}}(M)$ of leafwise holomorphic functions is not reduced to functions constant on the leaves and for any $g \in \mathcal{H}_{\mathcal{F}}(M)$, there exists $h \in \mathcal{H}_{\mathcal{F}}(M)$ such that $h - h \circ \gamma = g$. From the proof of this theorem we derive a foliated version of Mittag–Leffler Theorem.

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0 Introduction

Let E be a Fréchet space and $\gamma : E \longrightarrow E$ an automorphism. Denote by E^γ the set of elements of E invariant by γ; E^γ is the kernel of the (bounded) coboundary operator $\delta : f \in E \longmapsto (f - \gamma \cdot f) \in E$, thus it is a Fréchet subspace of E. The vector $g = \delta f$ is a 'measure' of the invariance defect of f. Suppose that we are given a bounded operator $T : E \longrightarrow E$ commuting with γ and we are interested to solve,

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in the subspace E^γ, the equation $Tf = g$ for given $g \in E^\gamma$. A natural way to do it is to solve firstly the equation in E (forgetting that g is γ-invariant) and correct a solution $f_0 \in E$ by adding an element h of $N =$ kernel of T to make the new solution $f = f_0 + h$ invariant by γ that is, satisfying the relation $\gamma \cdot (f_0 + h) = f_0 + h$ i.e. $h - \gamma \cdot h = \gamma \cdot f_0 - f_0$. (The element $(\gamma \cdot f_0 - f_0)$ is in N.) This brings us naturally to solve the following problem: *given $g \in N$, does there exist $h \in N$ such that $h - \gamma \cdot h = g$?* This is the *cohomological equation* of the 'dynamical system' (N, γ). (The terminology comes from the fact that the first cohomology space $H^1(\mathbb{Z}, N)$ of the discrete group $\mathbb{Z}$ with coefficients in the $\mathbb{Z}$-module N is exactly the cokernel of the operator $\delta : N \longrightarrow N$.)

This problem is a general version of different geometric situations. For instance, if M is a compact manifold, any diffeomorphism γ of M gives rise to an automorphism of the Fréchet space $E = \mathfrak{X}(M)$ of C^∞ vector fields; the cohomology group $H^1(\mathbb{Z}, \mathfrak{X}(M))$ contains exactly the infinitesimal deformations of the action of $\mathbb{Z}$ on M via the diffeomorphism γ (an explicit example of computation is given in [17]). Some other examples of cohomological equations were concretely investigated in [4]. Guichard's Theorem is also an illustration of the problem mentioned above. The group Γ generated by the affine transformation $\gamma : z \in \mathbb{C} \longmapsto z + 1 \in \mathbb{C}$ acts freely and properly and the quotient $M = \mathbb{C}/\Gamma$ is a non compact Riemann surface isomorphic to $\mathbb{C}^*$. Let E the Fréchet space of C^∞-functions on $\mathbb{C}$; then E^γ is the space of C^∞-functions on M. Now if T is the Cauchy–Riemann operator $\overline{\partial}_0 : h \in E \longmapsto \frac{\partial h}{\partial \overline{z}} \in E$, then solving the equation $\overline{\partial}_0 f = \phi$ in E^γ (this is nothing else the $\overline{\partial}$-problem on the complex manifold M) introduces automatically the cohomological equation $h - h \circ \gamma = g$ in the space $N = \mathcal{H}(\mathbb{C})$ of holomorphic functions on $\mathbb{C}$ which is the kernel of $\overline{\partial}_0 : E \longrightarrow E$. (But certainly this was not the motivation of Guichard in his paper [12] even if at that period the problem of solving the equation $\overline{\partial} f = \phi d\overline{z}$ on $\mathbb{C}^*$ was still open!) Guichard's Theorem with some type of growth conditions was studied in [2].

In this paper we are interested in the case where M is a differentiable manifold endowed with a complex simple foliation $\mathcal{F}$ whose leaves are non compact simply connected Riemann surfaces. Suppose that we are given a diffeomorphism γ of M which fixes each leaf F of $\mathcal{F}$ and such that the induced mapping $\gamma : F \longrightarrow F$ is a biholomorphism acting freely and properly (behaviour of γ is similar to the one of a translation on $\mathbb{C}$). This action induces an automorphism γ on the Fréchet space $\mathcal{H}_{\mathcal{F}}(M)$ of leafwise holomorphic functions (or *$\mathcal{F}$-holomorphic* functions). Under the hypothesis that the leaves are all parabolic or all hyperbolic, we prove that the vector space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to functions constant on the leaves and given any $g \in \mathcal{H}_{\mathcal{F}}(M)$ there exists $h \in \mathcal{H}_{\mathcal{F}}(M)$ such that $h - h \circ \gamma = g$. (This problem was already investigated for some other examples of complex foliations in [7].) Our proof follows from a sequence of lemmas. One of them uses arguments of cohomology of discrete groups and the others consist in solving the $\overline{\partial}$-problem along the leaves (or the $\overline{\partial}$-problem with parameter which was already considered in [5,10]). By the way, the proofs of the different lemmas contains some extra material which allows us to give for instance a foliated version of Mittag–Leffler Theorem.

Unless otherwise stated all the structures considered in this paper will be of class C^∞. Any open cover we consider will be *locally finite* that is, any compact set of M intersects only a finitely many open sets of this cover.

1 Complex foliations

Let M be a differentiable manifold of dimension $2m+n$ endowed with a codimension n foliation $\mathcal{F}$ (then the dimension of $\mathcal{F}$ is $2m$).

Definition 1.1 The foliation $\mathcal{F}$ is said to be ***complex*** if it can be defined by an open cover $\{U_i\}$ of M and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \longrightarrow U_i$ (where Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$) such that, for every pair (i, j) with $U_i \cap U_j \neq \emptyset$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \longrightarrow \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', t') = \left(\phi_{ij}^1(z, t), \phi_{ij}^2(t)\right)$ with $\phi_{ij}^1(z, t)$ holomorphic in z for t fixed.

An open set U of M like one of the cover $\mathcal{U}$ is called *adapted* to the foliation. Any leaf of $\mathcal{F}$ is a complex manifold of dimension m. The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold. A manifold M with a complex foliation $\mathcal{F}$ will be denoted $(M, \mathcal{F})$.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two complex foliations. A *morphism* from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$ is a differentiable mapping $f : M \longrightarrow M'$ which sends every leaf F of $\mathcal{F}$ into a leaf F' of $\mathcal{F}'$ such that the restriction map $F \xrightarrow{f} F'$ is holomorphic.

We say that a morphism $f : (M, \mathcal{F}) \longrightarrow (M', \mathcal{F}')$ is an *isomorphism* of complex foliations (*automorphism* of $(M, \mathcal{F})$ if $(M, \mathcal{F}) = (M', \mathcal{F}')$) if f is a diffeomorphism whose restriction to any leaf $F \longrightarrow F'$ (where $F' = f(F)$) is a biholomorphism. We say that two complex foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are *conjugated* if there exists an isomorphism $f : (M, \mathcal{F}) \longrightarrow (M, \mathcal{F}')$. Automorphisms of $\mathcal{F}$ form a group denoted $G(\mathcal{F})$.

Example 1.2 (i) Any complex manifold M of dimension m is a complex foliation of dimension m. Its automorphism group is exactly the automorphism group of the complex manifold M.

(ii) Any holomorphic foliation (on a complex manifold M) is a complex foliation.

(iii) Let B be a differentiable manifold and M an open set of $\mathbb{C}^m \times B$. For $t \in B$, $M^t = \{z \in \mathbb{C}^m : (z, t) \in M\}$ is an open set of $\mathbb{C}^m$ called the *section* of M along t. Sections of M are leaves of a complex foliation $\mathcal{F}$ of dimension m called the complex *canonical* foliation of M.

(iv) Let F be a complex manifold and B a differentiable one. Any locally trivial fibration $F \hookrightarrow M \longrightarrow B$ whose cocycle takes values in the automorphism group $\mathrm{Aut}(F)$ of (the complex manifold) F is a complex foliation, the fibres being the leaves. If the fibration is trivial i.e. $M = F \times B$, we say that $\mathcal{F}$ is a *complex product foliation*. In that case all the leaves are holomorphically equivalent. Suppose that $\mathcal{F}$ is a complex foliation on $M = F \times B$ whose leaves are the factors $F \times \{t\}$ but the complex structure may depend on t; then we say that $\mathcal{F}$ is a *differentiable product*.

(v) Let M be a Riemannian manifold equipped with an orientable foliation $\mathcal{F}$ by surfaces. To any vector $u \in T_y\mathcal{F}$ ($T_y\mathcal{F}$ is the tangent space to $\mathcal{F}$ at y) we associate the unique vector $v \in T_y\mathcal{F}$ with the same length and such that (u, v) is an orthonormal direct frame of the tangent space $T_y\mathcal{F}$ to the foliation $\mathcal{F}$. We set $J_{\mathcal{F}}u = v$; then $J_{\mathcal{F}}$ is an almost complex structure on the leaves. Since the dimension of the leaves is 2, $J_{\mathcal{F}}$ is integrable and then defines a complex structure which makes $\mathcal{F}$ a 1-dimensional complex foliation.

(vi) Let $(M, \mathcal{F})$ be a complex foliation. Suppose that $\mathcal{F}$ is Riemannian that is, the normal bundle $\nu\mathcal{F} = TM/T\mathcal{F}$ supports a Riemannian metric invariant along the leaves (cf. [16] or [15]) and that the leaves are closed. Then the leaf space $B = M/\mathcal{F}$ is an orbifold. In that case we say that $(M, \mathcal{F})$ is *complex simple foliation*. One can always equip the manifold M with a Riemannian metric which is transversely complete (geodesics which are orthogonal to $\mathcal{F}$ are complete). Let $\iota : M^\# \longrightarrow M$ be the G-principal bundle (where G is the orthogonal group $\mathrm{O}(n)$) of orthonormal frames transverse to $\mathcal{F}$. Using Molino's Theorem (cf. [15]), one easily obtain the following:

Proposition 1.3 *Let $(M, \mathcal{F})$ be a simple complex foliation. Then $\mathcal{F}$ lifts to a complex foliation $\mathcal{F}^\#$ on $M^\#$ such that*:

(i) $dim_{\mathbb{C}}\mathcal{F}^\# = dim_{\mathbb{C}}\mathcal{F}$;
(ii) *G acts on $M^\#$ by automorphisms of $\mathcal{F}^\#$*;
(iii) *$\mathcal{F}^\#$ is a locally trivial fibration $M^\# \longrightarrow B$ over a differentiable manifold B*;
(iv) *every leaf $F^\#$ of $\mathcal{F}^\#$ projects (by ι) on a leaf F of $\mathcal{F}$ and $\iota : F^\# \longrightarrow F$ is a holomorphic covering.*

In [14] Meersseman and Verjovsky constructed the first example of a complex foliation on the sphere $\mathbb{S}^5$ by complex surfaces following a question asked in [6] on the existence of codimension one complex foliations on any odd sphere $\mathbb{S}^{2n+1}$.

2 The $\overline{\partial}_{\mathcal{F}}$-cohomology

Let $(M, \mathcal{F})$ be a complex foliation of dimension m. Let $\Omega^{pq}(\mathcal{F})$ be the space of foliated differential forms of type (p, q) that is, differential forms on M which can be written in local coordinates adapted to the foliation $(z, t) = (z_1, \ldots, z_m, t_1, \ldots, t_n)$ (the foliation is defined by the differential system $dt_1 = \cdots = dt_n = 0$):

$$\alpha = \sum \alpha_{j_1\ldots j_p k_1\ldots k_q}(z, t) dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}$$

where $\alpha_{j_1\ldots j_p k_1\ldots k_q}$ is a C^∞-function on (z, t). Let $\overline{\partial}_{\mathcal{F}} : \Omega^{pq}(\mathcal{F}) \longrightarrow \Omega^{p,q+1}(\mathcal{F})$ be the Cauchy–Riemann operator along the leaves defined by:

$$\overline{\partial}_{\mathcal{F}}\alpha = \sum \left(\sum_{s=1}^{m} \frac{\partial \alpha_{j_1\ldots j_p k_1\ldots k_q}}{\partial \overline{z}_s}(z, t) d\overline{z}_s \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \right)$$

where $\frac{\partial}{\partial \overline{z}_s} = \frac{1}{2}\left\{\frac{\partial}{\partial x_s} + i\frac{\partial}{\partial y_s}\right\}$ with $z_s = x_s + iy_s$. It satisfies $\overline{\partial}_{\mathcal{F}}^2 = 0$, hence we have a differential complex $0 \longrightarrow \Omega^{p0}(\mathcal{F}) \stackrel{\overline{\partial}_{\mathcal{F}}}{\longrightarrow} \Omega^{p1}(\mathcal{F}) \stackrel{\overline{\partial}_{\mathcal{F}}}{\longrightarrow} \cdots \stackrel{\overline{\partial}_{\mathcal{F}}}{\longrightarrow} \Omega^{p,m-1}(\mathcal{F}) \stackrel{\overline{\partial}_{\mathcal{F}}}{\longrightarrow} \Omega^{pm}(\mathcal{F}) \longrightarrow 0$ called the $\overline{\partial}_{\mathcal{F}}$*-complex* of $(M, \mathcal{F})$; its homology $H_{\mathcal{F}}^{pq}(M)$ is called the *foliated Dolbeault cohomology* (or the $\overline{\partial}_{\mathcal{F}}$*-cohomology*) of the complex foliation $(M, \mathcal{F})$. It is locally trivial, i.e. we have the following lemma.

2.1 Foliated Dolbeault–Grothendieck Lemma

Let $x \in M$. Then there exists an open neighborhood U of x adapted to the foliation such that, for every $p = 0, \ldots, m$, $H_{\mathcal{F}}^{pq}(U) = 0$ for $q \geq 1$.

The proof is a straightforward adaptation to the parametric case of the classical one.

One can describe the cohomology $H_{\mathcal{F}}^{p*}(M)$ by using a sheaf which is analogous to the sheaf of germs of holomorphic p-forms on a complex manifold. A p-form α is said to be $\mathcal{F}$*-holomorphic*, if it is foliated, of type $(p, 0)$ and satisfies $\overline{\partial}_{\mathcal{F}}\alpha = 0$. Locally, a $\mathcal{F}$*-holomorphic* p-form can be written: $\alpha = \sum \alpha_{j_1 \ldots j_p}(z, t) dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ with $\alpha_{j_1 \ldots j_p}$ holomorphic on z.

Let $\mathcal{H}_{\mathcal{F}}^p$ be the sheaf of germs of $\mathcal{F}$-holomorphic p-forms on M and $\widetilde{\Omega}^{pq}(\mathcal{F})$ the sheaf of germs of differential forms of type (p, q) on $\mathcal{F}$; $\widetilde{\Omega}^{pq}(\mathcal{F})$ is a fine sheaf. Lemma 2.1 implies the following proposition.

Proposition 2.2 *The sequence* $0 \longrightarrow \mathcal{H}_{\mathcal{F}}^p \hookrightarrow \widetilde{\Omega}^{p0}(\mathcal{F}) \stackrel{\overline{\partial}_{\mathcal{F}}}{\longrightarrow} \cdots \stackrel{\overline{\partial}_{\mathcal{F}}}{\longrightarrow} \widetilde{\Omega}^{pm}(\mathcal{F}) \longrightarrow 0$ *is a fine resolution of* $\mathcal{H}_{\mathcal{F}}^p$. *So we have* $H^q(M, \mathcal{H}_{\mathcal{F}}^p) = H_{\mathcal{F}}^{pq}(M)$, *for* $p, q = 0, 1, \ldots m$.

If $n \geq 1$, this resolution is not elliptic; it is only elliptic along the leaves. Hence the cohomology $H^*(M, \mathcal{H}_{\mathcal{F}}^p)$ is not necessarily finite dimensional even if the manifold M is compact.

Any isomorphism of complex foliations $(M, \mathcal{F}) \stackrel{f}{\longrightarrow} (M', \mathcal{F}')$ induces an isomorphism $f^* : H^*(M', \mathcal{H}_{\mathcal{F}'}^p) \longrightarrow H^*(M, \mathcal{H}_{\mathcal{F}}^p)$. In particular $H^*(M, \mathcal{H}_{\mathcal{F}}^p)$ depends only on the complex conjugacy class of $\mathcal{F}$.

For $p = 0$, we denote $\mathcal{H}_{\mathcal{F}}$ the sheaf $\mathcal{H}_{\mathcal{F}}^0$; its sections over an open set U of M are $\mathcal{F}$-holomorphic functions on U; they form a complex vector space which we will denote by $\mathcal{H}_{\mathcal{F}}^0(U)$ and simply $\mathcal{H}(U)$ in case the codimension of $\mathcal{F}$ is zero, that is, M is a complex manifold and the foliation has just one leaf, M itself.

Let $p \in \mathbb{N}$. An open set U of M (with the induced foliation) is said to be *p-acyclic*, if $H^q(U, \mathcal{H}_{\mathcal{F}}^p) = 0$ for any $q \geq 1$. An open cover $\mathcal{U} = \{U_i\}$ is *p-acyclic* if, for any multi-index $(i_0, \ldots, i_s)$ of I, the open set $U_{i_0 \ldots i_s} = U_{i_0} \cap \cdots \cap U_{i_s}$ is p-acyclic. We can see easily by Lemma 2.1 that such open cover exists and, in addition, can be chosen locally finite. By Leray's Theorem (cf. [11]), $H^*(M, \mathcal{H}_{\mathcal{F}}^p) = H^*(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^p)$ for any locally finite p-acyclic open cover $\mathcal{U}$.

We have two ways for computing the $\overline{\partial}_{\mathcal{F}}$-cohomology of $\mathcal{F}$: using foliated differential forms of type (p, q) and the $\overline{\partial}_{\mathcal{F}}$ operator or a locally finite p-acyclic open cover $\mathcal{U}$ adapted to the foliation and Cěch method. Both of the two points of view will be interesting for our purpose.

Let us start with a simple example. Let F be a complex manifold of dimension m and B a differentiable manifold. We denote by $C^\infty(B)$ the complex vector space of complex C^∞ functions on B. The following proposition is easy to prove.

Proposition 2.3 *Suppose that $\mathcal{F}$ is defined by a locally trivial fibration $F \stackrel{\pi}{\longrightarrow} M \longrightarrow B$ (the cocycle is with values in the biholomorphism group of the complex manifold F). Then $H^{p*}_{\mathcal{F}}(M) = H^{p*}(F) \otimes C^\infty(B)$ where $H^{p*}(F)$ is the Dolbeault cohomology of the complex manifold F. In particular, $H^{p*}_{\mathcal{F}}(M) = 0$ for $* \geq 1$ if the the fibre F is a Stein manifold.*

3 $\mathcal{F}$-holomorphic functions in dimension 1

In this section, M will be a differentiable manifold of dimension $2 + n$ equipped with a complex foliation $\mathcal{F}$ of dimension 1. We suppose $p = 0$; $\Omega^{00}(\mathcal{F})$ is the algebra $C^\infty(M)$ of complex C^∞-functions on M.

3.1 The C^∞-topology on $\Omega^{0q}(\mathcal{F})$

Let $\mathcal{U} = \{(U, \varphi)\}$ be a countable atlas defining $\mathcal{F}$ and $\{K_r\}_{r\in\mathbb{N}}$ a countable family of compact sets covering M and such that any one of them is contained in a chart of $\mathcal{U}$. For multi-indices $k = (k_1, k_2)$ and $\ell = (\ell_1, \dots, \ell_n)$, let $|k| = k_1 + k_2$, $|\ell| = \ell_1 + \cdots + \ell_n$ and $D^{k\ell} = \frac{\partial^{|k|+|\ell|}}{\partial z^{k_1} \partial \bar{z}^{k_2}} \partial t_1^{\ell_1} \dots \partial t_n^{\ell_n}$. For $f \in C^\infty(M)$, we set $||f||_s^r = \max_{|k|+|\ell|\leq s} \left\{ \sup_{K_r} \left| D^{k\ell}(f) \right| \right\}$. The family of semi-norms $||\ \ ||_s^r$ (indexed by $s \in \mathbb{N}$ and $r \in \mathbb{N}^*$) defines a topology on $C^\infty(M)$ which is also associated to the distance $\delta(f, g) = \sum_{s,r} \frac{1}{2^{s+r}} \inf\left(1, ||f - g||_s^r\right)$ invariant by translation. It is independent on the atlas $\{(U, \varphi)\}$ and the family $\{K_r\}$. It makes $C^\infty(M)$ a Fréchet space; this is the C^∞-topology on $C^\infty(M)$. With respect to this topology, the subspace $\mathcal{H}_{\mathcal{F}}(M)$ of $\mathcal{F}$-holomorphic functions on M is closed.

The C^∞-topology on the space $\Omega^{01}(\mathcal{F})$ can be defined in the same way since any element $\alpha \in \Omega^{01}(\mathcal{F})$ can be written in a local chart (U, φ) as $\alpha = f d\bar{z}$ with $f \in C^\infty(U)$.

3.2 Zeros and poles

Let U be an open set of $\mathbb{C} \times \mathbb{R}$ with its canonical complex foliation $\mathcal{F}$. (The factor $\mathbb{R}$ can be replaced by any differentiable manifold.)

Let $f : U \longrightarrow \mathbb{C}$ be a $\mathcal{F}$-holomorphic function and let Z be the set of its zeros. The restriction of f to any leaf F is a holomorphic function; then, if $f : F \longrightarrow \mathbb{C}$ is not identically zero, $Z \cap F$ is a discrete set of F. So at a point of $Z \cap F$ where f does not vanish identically, $Z \cap F$ is 'transverse' to F.

We say that a function $f : U \longrightarrow \mathbb{C}$ is *$\mathcal{F}$-meromorphic*, if its restriction to any leaf is a meromorphic function. Let $\mathcal{P}$ be the set of poles of f; then, similarly to the case of zeros, the intersection of $\mathcal{P}$ with any leaf is a discrete set of F.

A $\mathcal{F}$-holomorphic function $f : U \longrightarrow \mathbb{C}$ (resp. $\mathcal{F}$-meromorphic) is simply C^∞ on U (resp. on $U \backslash \mathcal{P}$), we cannot say anything else on the structure of its zero set or its set of poles. We will just make some remarks in case Z and $\mathcal{P}$ are C^∞ manifolds. Such functions exist of course: for instance if $\varphi :]-\eta, \eta[\longrightarrow U \subset \mathbb{C}$ is a differentiable curve, then $f(z, t) = z - \varphi(t)$ is a $\mathcal{F}$-holomorphic function and admits elements of the set:

$$\Lambda = \{(z, t) \in \mathbb{C} \times \mathbb{R} : z = \varphi(t) \text{ with } t \in]-\eta, \eta[\}$$

as zeros and $g(t) = \frac{1}{f(z,t)}$ is $\mathcal{F}$-meromorphic and admits Λ as set of poles.

Now, let Σ be a small submanifold contained in U and transverse to $\mathcal{F}$; it can be considered as the graph of a map $z_0 : t \in]-\eta, \eta[\longmapsto z_0(t) \in \mathbb{C}$ of class C^∞. Let V be an open neighborhood of Σ such that each section V^t is a disc with centre $z_0(t)$. Then a C^∞-function $f : U \longrightarrow \mathbb{C}$, $\mathcal{F}$-holomorphic outside Σ admits a Laurent expansion:

$$f(z, t) = \sum_{k=0}^{\infty} a_k(t)(z - z_0(t))^k + \sum_{m=1}^{\infty} \frac{b_m(t)}{(z - z_0(t))^m}$$

where the coefficients a_k and b_m are given as usual by the integral formulaes:

$$a_k(t) = \frac{1}{2i\pi} \int_{\gamma_1^t} \frac{f(\xi, t)}{(\xi - z_0(t))^{k+1}} d\xi \quad \text{and} \quad b_m(t) = \frac{1}{2i\pi} \int_{\gamma_2^t} (\xi - z_0(t))^{m-1} f(\xi, t) d\xi;$$

γ_1^t and γ_2^t are respectively the smallest circle and the biggest circle of an annulus containing the point (z, t) in the section V^t of V. The point $(z_0, t_0) \in \Sigma$ is singular if at least one of the coefficients $b_m(t_0)$ is nonzero; if there exits $m_0 \geq 1$ such that $b_{m_0}(t_0) \neq 0$ and $b_m(t_0) = 0$ for $m > m_0$, we say that $(z_0, t_0) \in \Sigma$ is a *pole* of f of *order* m_0; if there exist infinitely many $b_m(t_0)$ which are not zero, we say that $(z_0, t_0) \in \Sigma$ is an *essential singularity*; if $a_0(t_0)$ and all the $b_m(t_0)$ are zero, we say that (z_0, t_0) is a *zero* of f; its *multiplicity* is by definition the smallest integer $n \geq 1$ such that $a_n(t_0) \neq 0$. Since all the a_k and b_m are C^∞-functions in t, if $(z_0, t_0) \in \Sigma$ is a singular point of f (i.e. a pole or an essential singularity) by continuity, there exists $\delta > 0$ such that, for $|t - t_0| < \delta$, the point $(z_0(t), t)$ is also singular. Then the singular set of f is an open transversal to $\mathcal{F}$. These remarks allow us to prove easily the following proposition.

Proposition 3.3 *Let $f : M \longrightarrow \mathbb{C}$ be a C^∞-function; suppose that f is $\mathcal{F}$-holomorphic outside a discrete union of small transversals Σ_j. Then*:

(i) *each Σ_j is open*;
(ii) *if each Σ_j is a point, f extends to a $\mathcal{F}$-holomorphic function on M.*

4 The foliated Guichard's theorem

Let us start with a definition. An open set of $\mathbb{C}$ is said to be a *crown* if it is of type $C(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ where $r \in \mathbb{R}$ and $R \in]0, +\infty]$. Open crowns of $\mathbb{C}$ are of six *types*:

(i) $C(r, R) = \mathbb{C}$ if $r < 0$ and $R = +\infty$;
(ii) $C(r, R)$ is a disc if $r < 0$ and $R < +\infty$;
(iii) $C(r, R)$ is a punctured disc if $r = 0$ and $R < +\infty$;
(iv) $C(r, R) = \mathbb{C}^*$ if $r = 0$ and $R = +\infty$;
(v) $C(r, R)$ is the complement of a closed disc if $r > 0$ and $R = +\infty$;
(vi) $C(r, R)$ is an annulus if $0 < r < R < +\infty$. Two annulus $C(r, R)$ and $C(r', R')$ are holomorphically equivalent if and only if $\frac{R}{r} = \frac{R'}{r'}$.

An open set M of $\mathbb{C} \times B$ (B is any differentiable manifold) equipped with its complex canonical foliation $\mathcal{F}$ is called *$\mathcal{F}$-crowned* if each leaf M^t is an open crown of $\mathbb{C}$.

Recall that a simply connected Riemann surface F is isomorphic to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (F is *elliptic*), the complex plane $\mathbb{C}$ (F is *parabolic*) or the open unit disc $\mathbb{D}$ (F is *hyperbolic*).

Theorem 4.1 *Let $(M, \mathcal{F})$ be a codimension n complex simple foliation whose leaves are simply connected non compact Riemann surfaces all parabolic or all hyperbolic. Let $\gamma : M \longrightarrow M$ be an automorphism of the complex foliation which fixes each leaf F and acts freely and properly on F. Then, the space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to functions constant on the leaves and for any $g \in \mathcal{H}_{\mathcal{F}}(M)$, the cohomological equation $h - h \circ \gamma = g$ admits a solution $h \in \mathcal{H}_{\mathcal{F}}(M)$ i.e. the vector space $H^1(\mathbb{Z}, \mathcal{H}_{\mathcal{F}}(M))$ is trivial (here $\mathcal{H}_{\mathcal{F}}(M)$ is viewed as a $\mathbb{Z}$-module via the action $(k, f) \in \mathbb{Z} \times \mathcal{H}_{\mathcal{F}}(M) \longmapsto f \circ \gamma^k \in \mathcal{H}_{\mathcal{F}}(M)$).*

The case B is a point, $F = \mathbb{C}$ and $\gamma(z) = z + 1$ is exactly Guichard's Theorem [12].

Proof The theorem will be a consequence of the following sequence of lemmas. Some of them will allow us to give a foliated version of Mittag–Leffler Theorem.

Lemma 1 *Consider the G-principal bundle $\iota : M^\# \longrightarrow M$ of orthonormal frames transverse to $\mathcal{F}$ with the lifted complex foliation $\mathcal{F}^\#$. Then $H^{0*}_{\mathcal{F}}(M)$ injects in $H^{0*}_{\mathcal{F}^\#}(M^\#)$.*

Proof Since $G = \mathrm{O}(n)$ acts on $M^\#$ by automorphisms of $\mathcal{F}^\#$, the foliated differential forms of type $(0, q)$ on M are the foliated differential forms on $M^\#$ of type $(0, q)$ which are invariant by G (they form a vector space denoted $\Omega^{0q}_G(\mathcal{F}^\#)$); then the cohomology $H^{0q}_{\mathcal{F}}(M)$ is canonically isomorphic to the cohomology $H^{0q}_{\mathcal{F}^\#, G}(M^\#)$ of the differential complex:

$$0 \longrightarrow \Omega^{00}_G(\mathcal{F}^\#) \xrightarrow{\overline{\partial}_{\mathcal{F}^\#}} \Omega^{01}_G(\mathcal{F}^\#) \xrightarrow{\overline{\partial}_{\mathcal{F}^\#}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}^\#}} \Omega^{0,m-1}_G(\mathcal{F}^\#) \xrightarrow{\overline{\partial}_{\mathcal{F}^\#}} \Omega^{0m}_G(\mathcal{F}^\#) \longrightarrow 0.$$

Let μ be a normalized Haar measure on the compact Lie group G. Then the averaging map $\sigma : \Omega^{0q}(\mathcal{F}^\#) \longrightarrow \Omega^{0q}_G(\mathcal{F}^\#)$ defined by $\sigma(\alpha) = \int_G g^*(\alpha) d\mu(g)$ is linear and continuous. It induces an injective linear map in cohomology $\sigma : H^{0*}_{\mathcal{F}}(M) = H^{0*}_{\mathcal{F}^\#,G}(M^\#) \hookrightarrow H^{0*}_{\mathcal{F}^\#}(M^\#)$. Then to prove $H^{0*}_{\mathcal{F}}(M) = 0$, it is sufficient to prove the nullity of $H^{0*}_{\mathcal{F}^\#}(M^\#)$. □

Lemma 2 *The cohomology vector space $H^{01}_{\mathcal{F}}(M)$ is zero and the space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to functions constant on the leaves.*

Proof From Lemma 1 one can assume that $\mathcal{F}$ is defined by a (differentiable) locally trivial fibration $\pi : M \longrightarrow B$ where B is a C^∞-manifold.

For any $t \in B$, the fibre $F_t = \pi^{-1}(t)$ is a simply connected non compact Riemann surface; then it is isomorphic to the complex plane $\mathbb{C}$ or the open unit disc $\mathbb{D}$. (From now on F will denote $\mathbb{C}$ or $\mathbb{D}$.)

(i) *Leaves are hyperbolic*

Any point $b \in B$ has an open neighborhood $V \subset B$ diffeomorphic to an open ball in $\mathbb{R}^n$ and such that $U = \pi^{-1}(V)$ is diffeomorphic to the product $F \times V$. By the uniformization theorem with parameter (cf. Ahlfors–Bers [1]), the complex foliation $(U, \mathcal{F})$ is isomorphic to the complex product $F \times V$. By proposition 1.3 we have $H^{01}_{\mathcal{F}}(U) = H^{01}(F) \otimes C^\infty(V) = 0$. (The Dolbeault cohomology $H^{0q}(F)$ of any non compact Riemann surface F is trivial for $q \geq 1$ [9].)

(ii) *Leaves are parabolic*

In that case uniformization theorem with parameter does not work. But one can solve the $\overline{\partial}_{\mathcal{F}}$ by the same procedure as in Lemma 3: it consists to take K_j such that K^t_j is a disc and Ω_j an open neighborhood of K_j such that Ω^t_j is also a disc. Then the series (L) is simply a Taylor series. Everything works.

Now let $\mathcal{V} = \{V_j\}_{j\in J}$ be an open cover of B whose elements V_j are diffeomorphic to a ball of $\mathbb{R}^n$. For any $j \in J$, the foliation $(U_j, \mathcal{F})$, where U_j is the open set $U_j = \pi^{-1}(V_j)$ of M, is isomorphic to the complex product $F \times V_j$ hence $H^{01}_{\mathcal{F}}(U_j) = 0$; furthermore $\mathcal{U} = \{U_j\}_{j\in J}$ is an open cover of M. Let $\{\overline{\rho}_j\}$ be a C^∞-partition of 1 on B subordinated to the cover $\mathcal{V}$; then $\rho_j = \overline{\rho}_j \circ \pi$ is a C^∞-partition of 1 on M subordinated to the cover $\mathcal{U}$. Let $\omega \in \Omega^{01}_{\mathcal{F}}(M)$; clearly ω and its restriction ω_j to U_j are $\overline{\partial}_{\mathcal{F}}$-closed. Then there exists $h_j \in \Omega^{00}_{\mathcal{F}}(U_j) = C^\infty(U_j)$ such that $\omega_j = \overline{\partial}_{\mathcal{F}} h_j$. Let $h \in \Omega^{00}_{\mathcal{F}}(M) = C^\infty(M)$ defined by $h = \sum_{j\in J} \rho_j h_j$. By the linearity and the continuity (with respect to the C^∞-topology) of the operator $\overline{\partial}_{\mathcal{F}}$, we have:

$$\overline{\partial}_{\mathcal{F}} h = \overline{\partial}_{\mathcal{F}} \left(\sum_{j\in J} \rho_j h_j \right) = \sum_{j\in J} \rho_j \overline{\partial}_{\mathcal{F}} h_j = \sum_{j\in J} \rho_j \omega_j = \omega.$$

(In the preceding sequence of equalities we used the fact $\overline{\partial}_{\mathcal{F}} \rho_j = 0$ because ρ_j is constant along the fibres of π.) This proves that $H^{01}_{\mathcal{F}}(M) = 0$.

Now, to prove that there are sufficiently $\mathcal{F}$-holomorphic functions (i.e. the vector space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to functions constant on the leaves) it is sufficient to see that locally, for instance on the open sets U_j, they abound and to glue them by the partition of the unity $\{\rho_j\}$ considered above. (This is always possible even the leaf space $B = M/\mathcal{F}$ is only an orbifold.) □

Lemma 3 *Let M be a crowned open set of $\mathbb{C} \times B$ where B is any differentiable manifold. Suppose that all the leaves M^t (which are crowns) are of the same type. Then the cohomology vector space $H^{01}_{\mathcal{F}}(M)$ is zero.*

Proof We have to prove that for any foliated 1-form $\omega = f(z,t)d\overline{z}$ of type $(0,1)$ (which is always $\overline{\partial}_{\mathcal{F}}$-closed) there exists a function $h \in C^\infty(M)$ such that $\overline{\partial}_{\mathcal{F}} h = \omega$.

For any $j \in \mathbb{N}$, let K_j be a subset of M such that, for any $t \in B$, the section K^t_j of K_j is a compact crowned subset of M^t ; let Ω_j be a $\mathcal{F}$-crowned open neighborhood of K_j. We suppose that $\overline{\Omega}_j$ is contained in the interior $\mathrm{int}(K_{j+1})$ of K_{j+1}. The sequence (K_j) is such that $\overline{\Omega}_j \subset \mathrm{int}(K_{j+1})$, is increasing and converges to M. Let $\varphi_j : \mathbb{C} \times B \longrightarrow \mathbb{R}$ be a C^∞-function with compact support in Ω_{j+1} and equal to 1 on Ω_j. We set:

$$\psi_j = \begin{cases} \varphi_j & \text{if } j = 0 \\ \varphi_j - \varphi_{j-1} & \text{if } j \geq 1. \end{cases}$$

For any $j \geq 1$, the function ψ_j is zero on Ω_{j-1} and the sequence (ψ_j) satisfies $\sum_{j=0}^{\infty} \psi_j = 1$. Let:

$$h_j(z,t) = \frac{1}{2i\pi} \int\limits_{\mathbb{C}} \frac{\psi_j(\xi,t) f(\xi,t)}{\xi - z} d\xi \wedge d\overline{\xi}.$$

The function h_j is well defined and C^∞ on M. By the Cauchy formula, we easily verify that $\overline{\partial}_{\mathcal{F}} h_j = \psi_j f$. Furthermore, for $j \geq 1$, ψ_j is zero on Ω_{j-1}, then h_j is $\mathcal{F}$-holomorphic on Ω_{j-1} (cf. [13]). Since Ω_{j-1} is $\mathcal{F}$-crowned, h_j admits a Laurent expansion on Ω_{j-1}:

$$h_j(z,t) = \sum_{k=0}^{\infty} a_{jk}(t) z^k + \sum_{m=1}^{\infty} \frac{b_{jm}(t)}{z^m} \tag{L}$$

where the coefficients a_{jk} and b_{jm} are given by the integral formulaes:

$$a_{jk}(t) = \frac{1}{2i\pi} \int\limits_{\gamma_1^t} \frac{h_j(\xi,t)}{\xi^{k+1}} d\xi \quad \text{and} \quad b_{jm}(t) = \frac{1}{2i\pi} \int\limits_{\gamma_2^t} \xi^{m-1} h_j(\xi,t) d\xi;$$

γ_1^t and γ_2^t are respectively the smallest circle and the biggest circle of the boundary of the compact set K^t_{j-1}. (In case K^t_{j-1} is a disc, γ_2^t is empty and $b_{jm}(t) = 0$ for $m \geq 1$.) The way the coefficients a_{jk} and b_{jm} are given shows that they are C^∞ in

t; on the other hand, the series (L) is given by the integral Cauchy formula, then it converges with respect to the C^∞-topology to the function h_j. Let $j \geq 1$; truncating conveniently the two sides (left and right) of the series (L), one gets a function:

$$v_j(z,t) = \sum_{k=0}^{k_j} a_{jk}(t)z^k + \sum_{m=1}^{m_j} \frac{b_{jm}(t)}{z^m}$$

which is $\mathcal{F}$-holomorphic on M and such that $\delta(h_j, v_j) < \frac{1}{2^j}$ (where δ is the distance on $C^\infty(M)$ defining the C^∞-topology). The series: $h_0 + \sum_{j=1}^{\infty}(h_j - v_j)$ converges with respect to the C^∞-topology to a function $h \in C^\infty(M)$. This function satisfies the equation $\overline{\partial}_{\mathcal{F}} h = f$; indeed, by the linearity and the continuity of $\overline{\partial}_{\mathcal{F}}$, one has:

$$\overline{\partial}_{\mathcal{F}} h = \overline{\partial}_{\mathcal{F}}\left(h_0 + \sum_{j=1}^{\infty}(h_j - v_j)\right) = \overline{\partial}_{\mathcal{F}} h_0 + \sum_{j=1}^{\infty}\overline{\partial}_{\mathcal{F}}(h_j - v_j) = \sum_{j=0}^{\infty}\psi_j f = f.$$

This ends the proof of the lemma. □

Lemma 4 *Let $M \xrightarrow{\pi} B$ be as in the statement of Theorem 4.1. Let $(\overline{M}, \overline{\mathcal{F}})$ be the complex foliation obtained as the quotient on $(M, \mathcal{F})$ by the action of the automorphism γ on M. Then the cohomology vector space $H^{01}_{\overline{\mathcal{F}}}(\overline{M})$ is zero.*

Proof From Lemma 1 one can also assume that $\mathcal{F}$ is defined by a (differentiable) locally trivial fibration $\pi : M \longrightarrow B$ where B is a C^∞-manifold.

The action of γ is free and proper and it leaves each fibre invariant. Then, the quotient manifold $\overline{M}$ is a differentiable locally trivial fibration $\overline{\pi} : \overline{M} \longrightarrow B$. Let V be an open set of B diffeomorphic to a ball of $\mathbb{R}^n$ and $U = \pi^{-1}(V)$; then U is diffeomorphic to $\overline{F} \times V$ where $\overline{F}$ is a crown of $\mathbb{C}$ (cf. [8]). The open set U with the complex foliation $\overline{\mathcal{F}}$ is isomorphic to a crowned open set of $\mathbb{C} \times B$ with its canonical complex foliation. By Lemma 2, we have $H^{01}_{\overline{\mathcal{F}}}(U) = 0$. Now, a partition of 1 argument like in the same Lemma allows us to conclude that $H^{01}_{\overline{\mathcal{F}}}(\overline{M}) = 0$. □

End of the proof of Theorem 4.1 Let us start with a more general fact. Let $\mathcal{F}$ be a complex foliation of dimension m on a manifold M and Γ a countable group which acts freely and properly by automorphisms of $(M, \mathcal{F})$. Then the quotient manifold $\overline{M} = M/\Gamma$ is endowed with the induced foliation $\overline{\mathcal{F}}$ in such way that the canonical projection $\pi : M \longrightarrow \overline{M}$ is a foliated covering map of complex foliations. The pull-back $\pi^*(\mathcal{H}_{\overline{\mathcal{F}}})$ of the sheaf $\mathcal{H}_{\overline{\mathcal{F}}}$ by π is exactly the sheaf $\mathcal{H}_{\mathcal{F}}$. Then there exists a spectral sequence whose E_2 term is:

$$E_2^{k\ell} = H^k(\Gamma, H^\ell(M, \mathcal{H}_{\mathcal{F}}))$$

and converging to $H^*(\overline{M}, \mathcal{H}_{\overline{\mathcal{F}}})$. Here $H^k(\Gamma, H^\ell(M, \mathcal{H}_{\mathcal{F}}))$ is the k-cohomology of the discrete group Γ with values in the Γ-module $H^\ell(M, \mathcal{H}_{\mathcal{F}})$ (Γ acts on $(M, \mathcal{F})$ by automorphisms so it acts on $H^\ell(M, \mathcal{H}_\mathcal{F})$). (See [3] for a construction of this spectral sequence.) If M is acyclic i.e.:

$$H^\ell(M, \mathcal{H}_\mathcal{F}) = \begin{cases} \mathcal{H}_\mathcal{F}(M) & \text{if } \ell = 0 \\ 0 & \text{if } \ell \geq 1 \end{cases}$$

the sequence E_r converges at the E_2 term and $H^k(\overline{M}, \mathcal{H}_{\overline{\mathcal{F}}}) = H^k(\Gamma, \mathcal{H}_\mathcal{F}(M))$ where the action of Γ on $\mathcal{H}_\mathcal{F}(M)$ is given by $(\gamma, f) \in \Gamma \times \mathcal{H}_\mathcal{F}(M) \longmapsto f \circ \gamma^{-1} \in \mathcal{H}_\mathcal{F}(M)$. But in our case $\Gamma = \mathbb{Z}$. So $\mathcal{H}_\mathcal{F}(M)/\mathcal{C} = H^1(\mathbb{Z}, \mathcal{H}_\mathcal{F}(M)) = H^1(\overline{M}, \mathcal{H}_{\overline{\mathcal{F}}}) = H^{01}_{\overline{\mathcal{F}}}(\overline{M}) = 0$ where $\mathcal{C}$ is the subspace of $\mathcal{H}_\mathcal{F}(M)$ generated by elements of the form $h - h \circ \gamma$. (The last equality is obtained by applying Lemma 4.) This shows that the coboundary operator $h \in \mathcal{H}_\mathcal{F}(M) \longmapsto (h - h \circ \gamma) \in \mathcal{H}_\mathcal{F}(M)$ is surjective that is, for any $g \in \mathcal{H}_\mathcal{F}(M)$ the cohomological equation $h - h \circ \gamma = g$ admits a solution $h \in \mathcal{H}_\mathcal{F}(M)$. □

Now we will use the results of the different steps of the proof of Theorem 4.1 to derive a parametric version of the Mittag–Leffler theorem.

In Sect. 3 we have defined the notion of $\mathcal{F}$-meromorphic function for a complex foliation $(M, \mathcal{F})$ whose leaves are Riemann surfaces. We have also seen that given a small submanifold Σ transverse to $\mathcal{F}$ ('small' means contained in an open set U adapted to the foliation), then there exists a $\mathcal{F}$-holomorphic function on U which admits Σ as zero set and a $\mathcal{F}$-meromorphic function which admits Σ as a set of poles. So the following question is natural: *Given a discrete countable union Σ of small submanifolds Σ_i transverse to $\mathcal{F}$, does there exist a global $\mathcal{F}$-meromorphic function on M which admits Σ as set of poles?* In other words is there a foliated (or a parameter) version of Mittag–Leffler Theorem? The answer is positive for the category of complex foliations we have considered and in case the family $\{\Sigma_i\}$ satisfies certain conditions.

Let $\mathcal{P}$ be a countable union of subsets P of M; we say that $\mathcal{P}$ is an *acyclic family*, if it satisfies the following conditions:

- $\mathcal{P}$ is discrete i.e. if $P, P' \in \mathcal{P}$ with $P \neq P'$, there exist two disjoint open sets V and V' adapted to $\mathcal{F}$ containing P and P' respectively;
- there exists an acyclic cover $\mathcal{U}$ by open sets adapted to the foliation such that any $P \in \mathcal{P}$ is contained in some $U \in \mathcal{U}$ and each $U \in \mathcal{U}$ contains at most one element $P \in \mathcal{P}$. We say that the cover $\mathcal{U}$ is *associated* to $\mathcal{P}$.

Theorem 4.2 (Mittag–Leffler with parameter) *Suppose that the complex foliation $(M, \mathcal{F})$ satisfies the hypotheses of Theorem 4.1 or Lemma 3. Let Σ be an acyclic family of submanifolds Σ_i transverse to $\mathcal{F}$ and $\mathcal{U} = \{U_i\}$ an acyclic open cover associated to Σ. Suppose that we are given on each U_i a $\mathcal{F}$-meromorphic function $h_i : U_i \longrightarrow \mathbb{C}$ which is $\mathcal{F}$-holomorphic outside Σ_i and that $h_i - h_j$ is $\mathcal{F}$-holomorphic on $U_i \cap U_j$. Then there exists a $\mathcal{F}$-meromorphic function $h : M \longrightarrow \mathbb{C}$ such that $h - h_i$ is $\mathcal{F}$-holomorphic on U_i.*

Proof The open cover $\mathcal{U} = \{U_i\}$ is acyclic; by Leray's Theorem, $H^1(\mathcal{U}, \mathcal{H}_\mathcal{F}) = H^1(M, \mathcal{H}_\mathcal{F})$. Theorem 4.1 implies $H^1(M, \mathcal{H}_\mathcal{F}) = 0$; then $H^1(\mathcal{U}, \mathcal{H}_\mathcal{F}) = 0$.

Let $f_{ij} = h_j - h_i$; we have $f_{jk} - f_{ik} + f_{ij}$ for any (i, j, k), then $\{f_{ij}\}$ is a 1-cocycle on $\mathcal{U}$ with values in the sheaf $\mathcal{H}_{\mathcal{F}}$. Since $H^1(\mathcal{U}, \mathcal{F}_{\mathcal{F}}) = 0$, $\{f_{ij}\}$ is a coboundary i.e. there exists a family of $\mathcal{F}$-holomorphic functions $f_i : U_i \longrightarrow \mathbb{C}$ satisfying $f_j - f_i = f_{ij} = h_j - h_i$ then $f_j - h_j = f_i - h_i$ on $U_i \cap U_j$ i.e. the collection $\{f_i - h_i\}$ defines a global $\mathcal{F}$-meromorphic function $h : M \longrightarrow \mathbb{C}$ which is $\mathcal{F}$-holomorphic outside the union $\cup_i \Sigma_i$. $\square$

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